Spectrum comparison in a system of reaction-diffusion equations with conservation property (Nonlinear Partial Differential Equations, Dynamical Systems and Their Applications)

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Abstract

We are dealing with a system of reaction-diffusion equations with a conservation of mass in a bounded domain with the Neumann boundary condition. The stationary problem of this system can be reduced to a scalar semilinear elliptic equation with a nonlocal term. We report some result for the spectrum comparison between the linearized eigenvalue problem of an equilibrium solution for the system and the corresponding linearized problem for the scalar nonlocal equation.

1 Introduction

We are concerned with the following two component system of reaction-diffusion equations in a bounded domain $\Omega \subset \mathbb{R}^N$ with the smooth boundary $\partial\Omega$:

$$\begin{cases} u_t = d\Delta u - g(u+\gamma v) + kv, \\ v_t = \Delta v + g(u+\gamma v) - kv \end{cases} \qquad (x \in \Omega,\ t > 0) \tag{1.1}$$

with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad (x \in \partial\Omega), \tag{1.2}$$

where d, k are positive constants and γ is a constant satisfying $0 \leq \gamma \leq 1$. Throughout the present article we assume that g is of class C^1 and $0 < d < 1$. This system has a conservation of mass by the property

$$\frac{d}{dt}\int_\Omega (u(x,t) + v(x,t))\,dx = 0$$

in a time interval on which the solution $(u(x,t), v(x,t))$ is defined. Thus we consider (1.1)-(1.2) with a constraint

$$s := \frac{1}{|\Omega|}\int_\Omega (u(x,t) + v(x,t))\,dx \qquad (t \geq 0). \tag{1.3}$$

We assume that the initial data of the solution $(u(x,t), v(x,t))$ are taken in an appropriate function space in which the C^1-semiflow is generated (for instance see [6]).

Moreover, this system possesses a Lyapunov function given by

$$\mathcal{E}_\gamma(u,v) := \int_\Omega \left\{ \frac{d}{2}|\nabla(u+\gamma v)|^2 + (1-d\gamma)G(u+\gamma v) + \frac{dk}{2}(u+\gamma v)^2 + \frac{k(1-\gamma)}{2(1-d)}(du+v)^2 + \frac{\gamma}{2}|\nabla(du+v)|^2 \right\} dx. \quad (1.4)$$

In fact, for the solution $(u(x,t), v(x,t))$

$$\frac{d}{dt}\mathcal{E}_\gamma(u(\cdot,t), v(\cdot,t)) = -\frac{1-d^2\gamma}{1-d\gamma}\int_\Omega |(u+\gamma v)_t|^2 dx - \frac{k(1-d\gamma)}{1-d}\int_\Omega |\nabla(du+v)|^2 dx - \frac{\gamma(1-\gamma)}{1-d\gamma}\int_\Omega |(du+v)_t|^2 dx \le 0$$

holds. This Lyapunov function is found in [9] and [7] for $\gamma = 0$ and $\gamma \in (0,1]$ respectively. By virtue of the Lyapunov function the omega-limit set of any bounded orbit in the phase space consists of equilibrium solutions (see [4]).

2 Spectrum comparison

Every equilibrium solution to (1.1)-(1.2) with (1.3) is obtained by solving the stationary problem

$$\begin{cases} d\Delta u - g(u+\gamma v) + kv = 0, \\ \Delta v + g(u+\gamma v) - kv = 0 \end{cases} \quad (x \in \Omega), \qquad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad (x \in \partial\Omega), \quad (2.1)$$

with the constraint

$$s = \overline{u+v} := \frac{1}{|\Omega|}\int_\Omega (u(x)+v(x))\,dx. \quad (2.2)$$

By putting $w = u + \gamma v$ the equations are transformed into

$$\begin{cases} d\Delta w - (1-d\gamma)g(w) + k(1-d\gamma)v = 0, \\ d\Delta w + (1-d\gamma)\Delta v = 0. \end{cases} \quad (2.3)$$

The second equation yields

$$dw + (1-d\gamma)v = c, \quad (2.4)$$

thus (2.3) turns a single equation for w as

$$d\Delta w - (1-d\gamma)g(w) - kdw + kc = 0,$$

where c is an unknown constant.

We note that if $\gamma = 0$, (2.4) becomes

$$d\overline{u} + \overline{v} = \overline{u} + \overline{v} + (d-1)\overline{u} = s - (1-d)\overline{u} = c,$$

because of $w = u$. In the sequel when $\gamma = 0$, the system (2.3) is reduced to

$$d\Delta u - g(u) - kdu + k(s - (1-d)\overline{u}) = 0. \tag{2.5}$$

On the other hand for $\gamma = 1$ we obtain

$$d\Delta w - (1-d)(g(w) - c_w) - kd(w - s) = 0, \qquad c_w := \overline{g(w)}. \tag{2.6}$$

We notice that (2.5) is the Euler-Lagrange equations of the energy functional

$$E_0(u) := \frac{d}{2}\|\nabla u\|^2 + \int_\Omega F_0(u)\,dx + \frac{k}{2}(s - (1-d)\overline{u})^2, \quad F_0(u) := \int^u g(\xi)d\xi + \frac{kd}{2}u^2, \tag{2.7}$$

in $L^2(\Omega)$ while so is (2.6) of the functional

$$E_1(w) := \frac{d}{2}\|\nabla w\|^2 + \int_\Omega F_1(w)\,dx, \quad F_1(w) := \int^w (1-d)g(\xi)d\xi + \frac{kd}{2}w^2, \tag{2.8}$$

in $X := \{w \in L^2(\Omega) : \overline{w} = s\}$ respectively, where $\|\cdot\|$ stands for the L^2 norm.

Let $u = u^*(x)$ and $w = w^*(x)$ be solutions to (2.5) and (2.6). Then

$$(u^*, v^*) = (u^*(x), s - du^*(x) - (1-d)\overline{u^*}) \tag{2.9}$$

and

$$(u^*, v^*) = \frac{1}{1-d}(dw^*(x) - ds, s - dw^*(x)) \tag{2.10}$$

give solutions to (2.1) for $\gamma = 0$ and $\gamma = 1$ respectively.

Now consider the linearized operators at $u^*(x)$ and $w^*(x)$ for (2.5) and (2.6) as

$$\mathcal{L}_0(\varphi) := -d\Delta\varphi + (g'(u^*(\cdot)) + kd)\varphi + k(1-d)\overline{\varphi},$$

and

$$\mathcal{L}_1(\varphi) := -d\Delta\varphi + \{(1-d)g'(w^*(\cdot)) + kd\}\varphi - (1-d)\overline{g'(w^*)\varphi},$$

with the domains

$$\mathcal{D}(\mathcal{L}_0) = \{\varphi \in H^2(\Omega) : \partial\varphi/\partial\nu = 0 \ (x \in \partial\Omega)\},$$

and

$$\mathcal{D}(\mathcal{L}_1) = \{\varphi \in H^2(\Omega) : \overline{\varphi} = 0, \quad \partial\varphi/\partial\nu = 0 \ (x \in \partial\Omega)\},$$

respectively. Corresponding to (2.9) and (2.10), we have the linearized operators

$$\mathcal{A}_0 \begin{pmatrix} \phi \\ \psi \end{pmatrix} := - \begin{pmatrix} d\Delta\phi - g'(u^*(\cdot))\phi + k\psi \\ \Delta\psi + g'(u^*(\cdot))\phi - k\psi \end{pmatrix},$$

and

$$\mathcal{A}_1 \begin{pmatrix} \phi \\ \psi \end{pmatrix} := - \begin{pmatrix} d\Delta\phi - g'(w^*(\cdot))(\phi+\psi) + k\psi \\ \Delta\psi + g'(w^*(\cdot))(\phi+\psi) - k\psi \end{pmatrix}$$

with the domain

$$\begin{aligned} &\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}_1) \\ &\quad = \{(\phi,\psi) \in H^2(\Omega) \times H^2(\Omega) : \overline{\phi+\psi} = 0, \ \ \partial\phi/\partial\nu = \partial\psi/\partial\nu = 0 \ (x \in \partial\Omega)\}. \end{aligned}$$

We note that the spectrum of the operator $\mathcal{L}_j$ for $j = 1, 2$, consists of real eigenvalues. It is also seen that every eigenvalue with non-positive real part of $\mathcal{A}_0$ or $\mathcal{A}_1$ is real (see [7] and [8]).

The number of all the negative eigenvalues with counting multiplicity for each linearized operator is called the Morse index, while the number of all the non-positive eigenvalues with counting multiplicity for the operator is called the augmented Morse index.

By [8] and [7] we have the following result:

Theorem 2.1 *For each $j = 0, 1$ the Morse index and the augmented Morse index of $\mathcal{L}_j$ agree with those of $\mathcal{A}_j$.*

As an application of this theorem any stable equilibrium solution $(u^*(x), v^*(x))$ to (1.1) for $\gamma = 0, 1$ in the case that Ω is a finite interval has a monotone profile such that both of $u^*(x), v^*(x)$ are monotone (constant or strictly monotone). Indeed, $u^*(x)$ (resp. $w^*(x)$) of (2.5) (resp. (2.6)) must be monotone if it is stable (see [5] or [10]). Considering (2.9) and (2.10), we get the monotonicity.

We remark that the spectral comparison idea appears in [1] where the Cahn-Hillard equation and the phase-field system are treated. The authors of [1] consider the linearized eigenvalue problems of those equations and compare them with certain scalar eigenvalue problems. On the other hand in the present study we compare the systems with the scalar ones with some nonlocal terms. By considering the nonlocal problems the nice correspondence between the Morse indices is shown as stated in Theorem 2.1.

The main tool for the proof of Theorem 2.1 is the min-max principle for the eigenvalue problem (see [2] and [3]). More precisely, in the case $\gamma = 0$, based on the research for the phase-field system in [1], we transform the linearized operator $\mathcal{A}_0$ to a self-adjoint operator and apply the mini-max principle repeatedly to compare the eigenvalues ([8]). On the other hand for the case $\gamma = 1$ we are not able to find such a transformation for $\mathcal{A}_1$. Nonetheless, by introducing an auxiliary eigenvalue problem with a parameter, which allows a variational setting, we compare eigenvalues of $\mathcal{L}_1$ with those of the auxiliary problem. Then, apply a continuation argument between the auxiliary problem and that of $\mathcal{A}_1$ (for the details see [7]).

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References

[1] P. W. Bates and P. C. Fife, Spectral comparison principles for the Cahn-Hilliard and phase-field equations, and time scales for coarsening, Physica D **43** (1990), 335-348.

[2] R. Courant and D. Hilbert, *Method of Mathematical Physics Vol I*, Wiley-Interscience, New York, 1953.

[3] E. B. Davies, *Spectral Theory and Differential Operators*, Cambridge University Press, 1995.

[4] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, no.25, AMS, 1988.

[5] M. E. Gurtin and H. Matano, On the structure of equilibrium phase transitions within the gradient theory of fluids, Quart. Appl. Math. **156** (1988), 301-317.

[6] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin-New York, 1981.

[7] S. Jimbo and Y. Morita, Lyapunov function and spectrum comparison for a reaction-diffusion system with mass conservation, preprint.

[8] Y. Morita, Spectrum comparison for a conserved reaction-diffusion system with a variational property, J. Appl. Anal. Computation, **2** (2012), 57-71.

[9] Y. Morita and T. Ogawa, Stability and bifurcation of nonconstant solutions to a reaction-diffusion system with conservation of a mass, Nonlinearity **23** (2010), 1387-1411.

[10] T. Suzuki, and S. Tasaki, Stationary Fix-Caginalp equation with non-local term, Nonlinear Anal. **71** (2009), 1329-1349.